Definition of a Group

A group G is a set along with an operation $*$ such that

1. (Closure) $\forall a, b \in G$, $a * b \in G$.
2. (Associativity) $\forall a, b, c \in G$, $(a*b)*c = a*(b*c)$.
3. (Identity) $\exists e \in G$ s.t. $\forall a \in G$, $a*e = e*a = a$.
4. (Inverses) $\forall a \in G$, $\exists b \in G$ s.t.
$$a*b = b*a = e.$$

If in addition, $(G,*)$ satisfies the commutative property (i.e $\forall a,b \in G$, $a*b = b*a$), then we say $(G,*)$ is an abelian group.

Examples.

1. S_4 = symmetric group on 4 letters
$= \{\text{permutations of } \{1,2,3,4\}\}$.

cycle notation $\sigma = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \mapsto \begin{pmatrix}3\\1\\2\\4\end{pmatrix} \equiv (1\ 3\ 2)$

mapping notation — cycle notation.

$\therefore\ \sigma(1) = 3$
$\sigma(4) = 4$

Facts: • Every permutation can be written as a product of disjoint cycles eg $(1,2)(3,4)$ or $(1,4)(3,2)$

FYI disjoint cycles commute.

• Every permutation can be written as a product of transpositions.
(a,b)

eg $\sigma = (1,2,3)(1,4)$ means $\sigma(x) = (1,2,3)[(1,4)(x)]$

As a product of disjoint cycles.

$\sigma = (1,4,2,3)$

As a product of transpositions

$\sigma = (1,2,3)(1,4)$

$\cancel{(1,2)(1,3)}$.

$(1,3)(1,2)$

$\Rightarrow \sigma = (1,3)(1,2)(1,4)$

$(1,3)(1,2)[1]$
$= (1,3)[2] = 2$
$(1,3)(1,2)[2]$
$= (1,3)[1] = 3$
$(1,3)(1,2)[3]$
$= (1,3)[3] = 1$
$(1,3)(1,2) = (1,2,3)$

In general

$(a_1, \dots, a_m) = (a_1, a_m)(a_1, a_{m-1}) \cdots (a_1, a_2)$

$\therefore$ any permutation can be written as a product of transpositions.

Question: Is this product unique?

No - e.g $(1,2,3,4) = (1,4)(1,3)(1,2)$
$= (1,2)(2,3)(3,4)$

$= (1,2,3)(1,4)$

$(2,3,4,1) = (2,1)(2,4)(2,3)(4,1)(4,1)(4,1)(4,1)$

$=$

An *even permutation* is a permutation that can be written as the product of an even number of transpositions.

An *odd permutation* is a permutation that can be written as an odd number of transpositions

Amazing Fact: This definition is well-defined.

Why is this true? In other words, if $\sigma \in S_n$ can be written as a product of an even # of transpositions then it can't be written as a product of an odd # of transpositions [and vice versa].

Possible Proof: Suppose

$$\sigma_1 \sigma_2 \cdots \sigma_{2m+1} = \tau_1 \tau_2 \cdots \tau_{2k}$$

transpositions.

$$\Rightarrow \underbrace{\sigma_1 \sigma_2 \cdots \sigma_{2m+1} \tau_{2k}^{-1} \tau_{2k-1}^{-1} \cdots \tau_2^{-1}\tau_1^{-1}}_{\text{odd \# of transp.}} = e$$